Closed points on schemes

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I learnt the argument below from Akhil Mathew's MathOverflow post.

Thm: Every non-empty quasi-compact scheme has a closed point.

Pf. Fix a non-empty quasi-compact scheme X. Note a non-empty closed subset $Z \subseteq X$ is a point iff Z does not contain a proper closed subset.

- $\Leftarrow$ Indeed Z with its reduced scheme structure contains an affine open (in Z) subset U. Since $Z \setminus U$ is closed in Z, $Z = U$. Since every affine scheme has a closed point, Z must be a point.
- $\Rightarrow$ direction is clear.

Now we prove that the quasi-compactness of X implies existence of a non-empty closed subset which does not have a proper closed subset. To this end, set

$$\Lambda = \{\text{non-empty closed subsets of } X\}.$$

For $a, b \in \Lambda$, define $b \geq a$ if $a \supseteq b$.

Given a chain $\Gamma \subseteq \Lambda$

$\bigcap_{a \in \Gamma} a$ must be non-empty.

Otherwise, $X = \bigcup_{a \in \Gamma} X \setminus a$. Since X is quasi-compact there is a finite collection $a_1, a_2, \dots a_n \in \Gamma$ such that

$$X = \bigcup_{i=1}^{n} X \setminus a_i.$$

Consequently $\bigcap_{i=1}^{n} a_i = \emptyset$. Since Γ is totally ordered, there is a biggest element (w.r.t $\geq$) among $\{a_1, \dots, a_n\}$. $\bigcap_{i=1}^{n} a_i = \emptyset$ implies that biggest element has to be an empty set contradicting the fact that elts of Λ are non-empty.

Since $\bigcap_{a \in \Gamma} a$ is non-empty and closed in X, $\bigcap_{a \in \Gamma} a \in \Lambda$.

Thus every chain in Λ has a maximum in Λ.

Λ is non-empty as $X \in \Lambda$.

By Zorn's lemma, we can pick a maximal elt Z in Λ. Then Z is a non-empty closed subset which does not have any proper closed subset, proving our Thm.

Rmk. When X is noetherian one can prove the analogue of the Thm above without using Zorn's lemma.

Prop. Let X be a quasi-compact scheme. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of open subsets of X such that $\bigcup_{\alpha \in \Lambda} U_\alpha$ contains all closed points of X. Then

$$X = \bigcup_{\alpha \in \Lambda} U_\alpha.$$

Pf. $Z := X \setminus \bigcup_{\alpha \in \Lambda} U_\alpha$ is a closed subset of X. Z is quasi-compact as X is. Any closed pt in Z is closed in X. So Z must be empty.

Rmk: Let P be a (local) property of a quasi-compact scheme X. Suppose P is open, i.e if P holds at $x \in X$, then there is a open nbhd U_x of x such that P holds at every $y \in U_x$. If such a P holds at all closed points, then by the Thm above this P holds for every $x \in X$.